NOTES ON EXISTENCE OF CLASSICAL SOLUTIONS OF SECOND ORDER ELLIPTIC EQUATIONS

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1. Introduction

Be aware that this is still work in progress. Any comment is more than welcome.

This notes are based strongly in the books of D. Gilbarg and N. S. Trudinger [3], L. Nirenberg [5], L. C. Evans [2]. The aim is to give a very quick introduction to a priori estimates to prove existence of solutions for elliptic second order equations. When we are studying the Laplace and Poisson's problems, one obtain important inequalities regarding to the solutions, for instance the maximum principle for harmonic functions, and eventually estimates involving higher order derivatives. For more general equations the maximum principle can be obtained avoiding the potential theory, and we can get similar inequalities. From topology of Banach spaces and functional analysis it is possible to get existence results for solutions of the Dirichlet problem based on these inequalities. In the case of linear equations, the Schauder estimates implies the existence results for a wide variety of problems depending on the properties of the coefficients of the differential operator. Then by means of the continuity method, one can establish the invertibility of such operator from a known one. For quasilinear equations, the problem is more complicated, and topological methods like degree theory and fixed point results are extended to infinite dimensional Banach spaces. The Leray-Schauder fixed point theorem in Banach spaces gives the existence theory for the Dirichlet problem of a large class of quasilinear equations. Finally we say something about the fully nonlinear case and how continuity method can be also applied if certain a priori estimates are established.

This notes are not self-contained and there are many different results regarding the influence of the geometry of the domain Ω and its boundary $\partial\Omega$ to the solution of the Dirichlet problem that are not mentioned.

We will study second order partial differential equations that can be linear, or not, depending on some properties of its coefficients. For instance let $\Omega \subset \mathbb{R}^n$ be a domain (open and connected), $\mathbb{R}^{n\times n}$ the set of all square symmetric matrices n by n. A second order equation can be written in general as a function $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$

$$F[u] = F(x, u, Du, D^2u) = 0. \tag{1}$$

In case that the coefficients of highest order depend only on $(x, u, Du) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, then the equation is called **quasi-linear**, and can be written as

$$Qu = a^{ij}(x, u, Du)D_{ij}u + b(x, u, Du). \tag{2}$$

A quasi-linear equation such that the highest order term is linear is called **semi-linear**. If all coefficients are depending only on $x \in \Omega$ we refer to it as a **linear** equation

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u. \tag{3}$$

A **fully nonlinear** equation is a second order equation F that cannot be written as a quasi-linear equation.

Quasilinear equations are called **elliptic** if the coefficient matrix $[a^{ij}]$ is positive definite, more over, if λ and Λ denote the minimum and maximum eigenvalues of $[a^{ij}]$, then the equation is elliptic if $\lambda \geq 0$, **strictly elliptic** when $\lambda \geq \epsilon > 0$, and **uniformly elliptic** Λ/λ is bounded. A fully nonlinear equation F is elliptic if the matrix $F_{ij} = \partial F/\partial u_{ij}$ is positive definite.

The **Hölder space** $C^{k,\alpha}(\bar{\Omega})$, with exponent $0 < \alpha < 1$ is the Banach space of functions f with norm

$$|f|_{C^{k,\alpha}(\Omega)} = |f|_{C^k(\Omega)} + \max_{|j|=k} |D^j f|_{C^\alpha(\Omega)}, \tag{4}$$

where

$$|f|_{C^k(\Omega)} = \sum_{j=0}^{k} \max_{|r|=j} \sup_{x\in\Omega} |D^r f| \quad ; \quad |f|_{C^\alpha(\Omega)} = \sup_{\substack{x\neq y,\\ x,y\in\Omega}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}. \tag{5}$$

Hölder spaces are Banach spaces with the norm (4) when functions are defined in an open bounded set Ω.

Remark. The following semi-norms are also used:

Non-dimensional norms: If Ω is bounded and $d = \operatorname{diam}\Omega$ we define:

$$|f|'_{C^k(\Omega)} = \sum_{j=0}^{k} d^j \max_{|r|=j} \sup_{x\in\Omega} |D^r f| \quad ; \quad |f|'_{C^{k,\alpha}(\Omega)} = |f|'_{C^k(\Omega)} + d^{k+\alpha} \max_{|j|=k} |D^j f|_{C^\alpha(\Omega)}. \tag{6}$$

Interior norms: $d_x = \operatorname{dist}(x, \partial\Omega)$ and $d_{x,y} = \min\{d_x, d_y\}$:

$$\begin{gathered} |f|^*_{C^k(\Omega)} = \sum_{j=0}^{k} \max_{|r|=j} \sup_{x\in\Omega} d_x^j |D^r f| \quad ; \quad [f]^*_{C^{k,\alpha}(\Omega)} = \sup_{x\neq y; |\beta|=k} d_{x,y}^{k+\alpha} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha}. \\ |f|^*_{C^{k,\alpha}(\Omega)} = |f|^*_{C^k(\Omega)} + [f]^*_{C^{k,\alpha}(\Omega)}. \end{gathered} \tag{7}$$

Arzela-Ascoli Theorem. This is a fundamental result in mathematical analysis. It gives the necessary and sufficient conditions for a sequence of functions to have a uniformly convergent subsequent.

Theorem 1 (([1],appx. D)). *Suppose that $\{f_k\}_{k=1}^\infty$ is a sequence of functions defined on a domain $\Omega \subset \mathbb{R}^n$ such that*

(1) It is uniformly bounded, i.e. there is a constant $M > 0$ such that $|f_k(x)| \leq M$ for all $x \in \Omega$ and all k.

(2) It is uniformly equicontinuous*: for each $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f_k(x) - f_k(y)| < \epsilon$, for $x, y \in \Omega$ and all k.*

Then there exists a sub-sequence $\{f_{k_j}\}_{j=1}^\infty \subset \{f_k\}_{k=1}^\infty$ and a continuous function f such that $f_{k_j} \to f$, uniformly on compact subsets of Ω.

Some of the applications of the Arzela-Ascoli theorem we will be using are presented in the following examples.

Example 2. *Let $\{f_k\}$ be a uniformly bounded sequence of differentiable functions defined in a compact set $K \subset \mathbb{R}^n$. If $\{Df_k\}$ is uniformly bounded by $M > 0$, then by the mean value theorem we have*

$$|f_k(x) - f_k(y)| \leq \sup_{z\in K} |Df_k(z)|\, |x-y| \leq M|x-y|. \tag{8}$$

Note that this proves f_k to be actually a sequence of Lipschitz functions with the same Lipschitz constant. This is equivalent to equicontinuity of the family. Then by the Arzela-Ascoli theorem the sequence f_k has a subsequence that converges uniformly on compact sets. The limit function is also Lipschitz with same constant M.

Example 3. *Let $\{f_k\}$ be a uniformly bounded sequence of continuous functions defined in a compact set $K \subset \mathbb{R}^n$ and such that $f_k \in C^\alpha(K)$ for all k, and even more, there is a positive constant $M > 0$ such that $|f_k|_{C^\alpha(K)} \leq M$, for all k. Then again, this is equivalent to equicontinuity of the family and by the Arzela-Ascoli theorem the sequence f_k has a subsequence that converges uniformly on compact sets.*

2. Ellipticity

Consider the following example of partial differential operator in $\mathbb{R}^2$:

$$F = a_{11}u_{11} + a_{12}u_{12} + a_{22}u_{22} + b_1u_1 + b_2u_2 + cu,$$

then, the matrix F^{ij} is given by

$$(F^{ij}) = \begin{pmatrix} a_{11} & \frac{a_{12}}{2} \\ \frac{a_{12}}{2} & a_{22} \end{pmatrix},$$

hence, F is elliptic if

$$\det(F^{ij}) = a_{11}a_{22} - \frac{a_{12}^2}{4} \geq 0,$$

or equivalently

$$4a_{11}a_{22} - a_{12}^2 \geq 0.$$

It is not a trivial matter to generalise the concept of ellipticity to a higher order system of N equations for N functions $u_1, \ldots, u_N$. Here we present the notion of non-characteristics and how it can help us to understand ellipticity from this point of view.

In general one can consider the system of equations

$$F_i\left(x, u_1, \ldots, u_N, \frac{\partial^k u_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}, \ldots\right) = 0,$$

with $i, j = 1, 2, \ldots, N$, and $k = k_1 + \cdots + k_n$ where $1 \leq k \leq n_j$, that is, for each function u_j there is a highest order derivative n_j appearing in the system.

The semilinear case. Consider the case where each F_i is linear in the highest order derivative, then we can write the i - th equation as

$$\sum_{j=1}^{N} \sum_{k_1+\ldots+k_n=n_j} a_{ij}^{k_1\cdots k_n} \frac{\partial^{n_j} u_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} + G_i\left(x, u_1, \ldots, u_N, \frac{\partial^m u_j}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}, \ldots\right) = 0, \tag{9}$$

here, $1 \leq m \leq n_j - 1$.

Definition 4. *Let $p \in \mathbb{R}^n$, and S be a smooth hypersurface containing p. Suppose that $u_1, \ldots u_N$ are functions satisfying the system of semilinear system of equations* (9). *Suppose that the values of the functions u_j and the values of their derivatives up to order $n_j - 1$ are known on the surface in a neighbourhood around p. If we can calculate the n_j -th order derivatives of u_j at the point p, then the surface S is called* ***non-characteristic*** *at p with respect to the semimlinear system.*

Suppose that $y_1, \ldots, y_n$ are new coordinates around the point p, such that, around a neighbourhood of p, the surface S is given by $y_1 = 0$.

Observe that, since for the function u_j we know its $n_j - 1$ derivatives, then it is possible to obtain the n_j -th derivative in directions which are tangent to the surface.

Then the problem is now to compute the n_j -th order derivatives $\frac{\partial^{n_j} u_j}{\partial y_1^{n_j}}$. By the chain rule one sees that

$$\frac{\partial^{n_j} u_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} = \frac{\partial^{n_j} u_j}{\partial y_1^{n_j}} \left(\frac{\partial y_1}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial y_1}{\partial x_n}\right)^{k_n} + \begin{array}{c}\text{derivatives of order } n_j \\ \text{mixed with} \\ \text{tangent directions}\end{array} + \text{lower order terms.} \tag{10}$$

Making use of the equationg (9), we obtain

$$\sum_{j=1}^{N} \sum_{k_1+\ldots+k_n=n_j} a_{ij}^{k_1\cdots k_n} \frac{\partial^{n_j} u_j}{\partial y_1^{n_j}} \left(\frac{\partial y_1}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial y_1}{\partial x_n}\right)^{k_n} + \cdots + G_i = 0. \tag{11}$$

These equations can be solved for $\frac{\partial^{n_j} u_j}{\partial y_1^{n_j}}$ if and only if the determinant of the $N \times N$ matrix

$$m_{ij} = \sum_{k_1+\ldots+k_n=n_j} a_{ij}^{k_1\cdots k_n} \left(\frac{\partial y_1}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial y_1}{\partial x_n}\right)^{k_n}, \tag{12}$$

is different from zero at the point $p \in S$.

Recall that around $p \in S$, the surface S is the level set $y_1^{-1}\{0\}$, hence, the gradient

$$Dy_1(p) = \left(\left.\frac{\partial y_1}{\partial x_1}\right|_p, \left.\frac{\partial y_1}{\partial x_2}\right|_p, \ldots \left.\frac{\partial y_1}{\partial x_n}\right|_p \right) \tag{13}$$

is a multiple of the unit vector $\nu_p := (\nu_1, \ldots, \nu_n)$ normal to the surface S at p.

The *characteristic determinant* is the function $H : \mathbb{R}^n \to \mathbb{R}$ given by

$$H(x) = \det\left(m_{ij}(x)\right) = \det\left(\sum_{k_1+\ldots+k_n=n_j} a_{ij}^{k_1\cdots k_n} x_1^{k_1} \cdots x_n^{k_n} \right), \tag{14}$$

and the equation $H(x) = 0$ is called the *characteristic equation.*

Definition 5. *S is said characteristic at p if the characteristic equation is satisfied by the normal $\nu = (\nu_1, \ldots, \nu_n)$ to S at p, and it will be simply called a characteristic surface if it is characteristic at every point.*

Then the system (9) is called elliptic is the characteristic equation has no real solution other than the trivial, i.e., $H(x) = 0$ only for $x = 0$.

3. Maximum principle for the Laplace equation

This is one of the most important and fundamental results of partial differential equations of the elliptic type. There are many important properties of PDE's that follows if the maximum principle holds and the idea is the following: consider a smooth function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. The symmetric matrix of second order derivative has important information, for instance, one knows that if there is a local maximum at a point $x_0 \in \Omega$ then the Hessian matrix $D^2u(x_0)$ is negative definite, and the graph of u looks locally concave, while if x_0 is a local minimum the hessian matrix $D^2u(x_0)$ is positive definite, and the graph of u in this case looks locally convex. Since $\Delta u = \mathrm{Tr}(Du^2)$, the condition $\Delta u > 0$ in Ω suggests that the graph of the function u looks (in some sense) convex.

Proposition 6. *If $\Delta u > 0$ in Ω, then u attains its maximum value at the boundary $\partial\Omega$.*

Proof. If there is a local maximum of $u \in C^2(\Omega)$ at x_0 in the interior of Ω we would have $\frac{\partial^2 u}{\partial x_i^2}(x_0) \leq 0$ for all $i = 1, 2, \ldots, n$, which leads to a contradiction. □

Proposition 7. *If $\Delta u \geq 0$ in Ω, then u attains its maximum value at the boundary $\partial\Omega$. Further more, for any ball $\overline{B_r(x_0)} \subset \Omega$ it holds that $u(x_0) \leq \max_{x \in \partial B_r(x_0)} u(x)$.*

Proof. On the contrary lets assume there is $r > 0$ such that $\overline{B_r(x_0)} \subset \Omega$ and $u(x_0) > m$, where $m := \max_{x\in\partial B_r(x_0)} u(x)$. Now define the function

$$U(x) := u(x) + \frac{\alpha}{r^2}|x - x_0|^2, \tag{15}$$

where $\alpha > 0$ is a constant to be determine later. Note that $U(x_0) = u(x_0)$. On the other hand if $|x - x_0| = r$ we have

$$U(x) = u(x) + \alpha < m + \alpha. \tag{16}$$

Choose any $0 < \alpha < u(x_0) - m$ and then for $|x - x_0| = r$ it holds

$$U(x) < u(x_0). \tag{17}$$

We also note

$$\Delta U(x) = \Delta u(x) + \frac{2\alpha n}{r^2} > 0, \tag{18}$$

which contradicts the previous proposition. □

4. The Poisson's equation in domains of $\mathbb{R}^n$: Kellog's Theorem.

The most important elliptic equation is the Laplace equation $\Delta u = 0$. The solutions of this equation are called harmonic functions. When the right hand side is different from zero, $\Delta u = f$, it is called Poisson's equation.The following points outline one path to establish the solvability of the Laplace and Poisson's equations using the so called Perron's method in bounded domains of the Euclidean space:

- First we will start with Poisson's equation in the unbounded $\mathbb{R}^n$ by means of the Newton's kernel.
- After learning some properties of the Newton kernel and convolution with functions, we obtain the Green's representation formula. As an application we also obtain the Green's function in a ball for the Laplace equation.
- The solution of the Laplace equation in the ball is given explicitly by the Poisson's integral formula.
- Then we stress that in the ball, a solution of the Laplace equation, has certain regularity.
- To solve the Laplace equation in bounded domains Ω we make use of Perron's method for subharmonic functions.
- Then the solvability of the Dirichlet problem for Laplace equation regarding the boundary data is addressed.
- Poission's equation in bounded domains will follow from Laplace equation, using suitable function in Newton's kernel, and Perron's method with the corresponding boundary data.
- We keep developing estimates for the solutions and establish Kellogg's theorem. This kind of a priori estimates will be further developed in the next section to establish existence of solutions for linear equations avoiding potential theory, i.e., by obtaining a priori estimates.

We are basically using the so called Potential theory in the ball, and then Perron's method to get a solution with certain regularity. This is also the underlying idea for proving the existence of a solution in the *viscosity* sense for fully nonlinear equations.

We start by considering the Poission's equation in $\mathbb{R}^n$ namely

$$\Delta u = f. \tag{19}$$

For compactly supported functions f, an explicit solution can be obtained using the Newton's kernel

$$\Gamma(x) = \begin{cases} c|x|^{2-n}, & n \geq 3 \\ -c \log |x|, & n = 2 \end{cases}, \tag{20}$$

and then the solution is given by the convolution:

$$u(x) = (\Gamma * f)(x) := \int \Gamma(x-y) f(y) dy. \tag{21}$$

Note that the convolution is commutative $\Gamma * f = f * \Gamma$, and if additionally f is smooth and has compact support, we have the following properties:

(i) $\Gamma * \Delta f = f$,
(ii) $\Delta(\Gamma * f) = f$.

Recall integration by parts over a domain Ω in $\mathbb{R}^n$ for smooth function u and a vector field F, and the Green's theorem

$$\begin{aligned} \int_\Omega u \operatorname{div} F &= \int_\Omega \operatorname{div}(uF) - \int_\Omega F\cdot\nabla u, \\ &= \int_{\partial\Omega} uF\cdot\nu\, ds - \int_\Omega F\cdot\nabla u, \end{aligned} \tag{22}$$

where ν is the outward unit vector, ∇u is the gradient vector and $\cdot$ denotes the usual inner product in $\mathbb{R}^n$. Then the identity $\Gamma * \Delta f = f$ follows after integrating by parts in the complement of a small neighbourhood of the origin and then taking the limit when the radius of this ball goes to zero. The second identity follows from the first one and noticing that we can move the Laplace operator Δ inside the integral with respect to the variable that does not appear as an argument of Γ.

Using integration by parts and using limit properties of the Newton kernel, one can obtain the *Green's representation formula*, for all $y \in \Omega$:

$$u(y) = \int_{\partial\Omega}\left(u\frac{\partial\Gamma}{\partial\nu}(x-y) - \frac{\partial u}{\partial\nu}\Gamma(x-y)\right)ds + \int_\Omega \Gamma(x-y)\Delta u dx. \tag{23}$$

More over, suppose $h \in C^1(\bar\Omega)\cap C^2(\Omega)$ is harmonic in Ω. $G = \Gamma + h$ and suppose $G = 0$ on $\partial\Omega$, then it is possible to obtain the Green's function of the first kind for the domain Ω:

$$u(y) = \int_{\partial\Omega} u\frac{\partial G}{\partial\nu}ds + \int_\Omega G\Delta u dx. \tag{24}$$

Green's function on the ball B_R with centre at the origin can be determined using the inversion through the sphere of radius R, given by $i(y) = \frac{R^2}{|y|^2}y$. Note that

$$\frac{|y|^2}{R^2}|x - i(y)|^2 = \frac{|x|^2|y|^2}{R^2} - 2x\cdot y + R^2. \tag{25}$$

Then the Green function is given by

$$G(x,y) = \begin{cases} \Gamma(|x-y|) - \Gamma\left(\frac{|y|}{R}|x - i(y)|\right), & y \neq 0 \\ \Gamma(|x|) - \Gamma(R), & y = 0. \end{cases} \tag{26}$$

Note that the **Poisson's kernel** is the normal derivative of G on the boundary ∂B_R

$$\frac{\partial G}{\partial\nu} = \frac{R^2 - |y|^2}{n\omega_n R}\frac{1}{|x-y|^n}, \qquad x \in \partial B_R \tag{27}$$

There is an explicit solution in the case Ω is the B_R with centre in the origin:

$$u(x) = \int_{\partial B_R} K(x,y)\varphi(y)ds + \int_{B_R} G(x,y)f(y)dy, \tag{28}$$

where K is the Poisson kernel and G is the green function of the ball, and moreover, whenever φ is continuous in ∂B_R, then $u(x) \in C^0(\bar B_R)\cap C^2(B_R)$.

Theorem 8 (([3],T2.6))**.** *Let $B = B_R(0)$ and φ a continuous function on ∂B. Then the function u given by*

$$u(x) = \begin{cases} \frac{R^2-|x|^2}{n\omega_n R}\int_{\partial B}\frac{\varphi(y)}{|x-y|^n}ds_y, & x \in B \\ \varphi(x), & x \in \partial B \end{cases}, \tag{29}$$

is harmonic and $u \in C^0(\bar B)\cap C^2(B)$.

Poisson's kernel is smooth for all $x \in B$, then the smoothness of u will follow from the fact that we can differentiate under the integral. This is a consequence of the Lebesgue' Dominated Convergence Theorem in measure theory: Let (Ω,μ) a measure space, and $X \subseteq \mathbb{R}$ open. Consider a function $f : X\times\Omega \to \mathbb{R}$, such that for every $x \in X$, the function $f(x,\cdot) : \Omega \to \mathbb{R}$ is a Lebesgue-integrable function; for all $x \in X$ the derivative $\partial_x f$ exists for almost all $\omega \in \Omega$; There is a Lebesgue integrable function $g : \Omega \to \mathbb{R}$ such that $|\partial_x f(x,\omega)| \leq g(\omega)$, for all $x \in X$ and for almost every $\omega \in \Omega$. Then applying the Dominated Convergence Theorem, for all $x \in X$ we have:

$$\frac{d}{dx}\int_\Omega f(x,\omega)d\mu = \int_\Omega \partial_x f(x,\omega)d\mu. \tag{30}$$

Let $\Omega \in \mathbb{R}^n$ a domain. $u \in C^0(\Omega)$ is called **subharmonic** if its values are less or equal than the values of any harmonic function with bigger boundary data than u, all when restricted to any properly contained ball in Ω, more precisely, if for every ball B such that $\bar{B} \subset \Omega$ and every function h harmonic in B such that $u \leq h$ on ∂B then we also have $u \leq h$ in B. The definition of **superhamonic** functions follows by replacing $\leq$ by $\geq$. Now we list some properties of subharmonic functions

- If u is subharmonic in Ω then it satisfies the strong maximum principle (if maximum is attained in the interior then it is constant).
- When comparing a superharmonic function v and a subharmonic function u in the a bounded domain Ω such that $v \geq u$ on $\partial\Omega$ then if $v \neq u$ we have $v \geq u$ in all Ω.
- Let u be subharmonic in Ω and B such that $\bar{B} \subset \Omega$. By the Possion's integral of u in ∂B we obtain a harmonic function $\bar{u}$ in B such that $\bar{u} = u$ on ∂B. From this, we can construct the following function U, which is subharmonic in Ω given by

$$U(x) = \begin{cases} \bar{u}(x), & x \in B \\ u(x), & x \in \Omega \setminus B. \end{cases} \tag{31}$$

- Let $\{u_1, \ldots, u_m\}$ be a finite collection of subharmonic functions in Ω. Then $u(x) = \max_i\{u_i(x)\}$ is subharmonic.

The corresponding results for superharmonic functions follow from replacing u by $-u$ in each property.

Let Ω be a bounded domain and φ a function in Ω that is bounded in $\partial\Omega$. Then a $C^0(\bar{\Omega})$ function u that is subharmonic in Ω is called **subfunction** relative to φ, if $u \leq \varphi$ on $\partial\Omega$. We define **superfunction** in a similar way. Note that constant functions with value less or equal than $\inf_{\partial\Omega}\varphi$ are subfunctions. Also note that by the maximum principle every subfunction is less or equal that every superfunction.

Theorem 9 (Perron's Method ([3],T2.12))**.** *Let S_φ the set of subfunctions relative to the function $\varphi : \Omega \to \mathbb{R}$, where Ω is a bounded domain of $\mathbb{R}^n$ and φ is bounded on $\partial\Omega$. Then the function*

$$u(x) = \sup_{v \in S_\varphi} v(x), \tag{32}$$

is harmonic in Ω.

One way to prove Perron's methods is via Harnack's convergence theorem using Harnack's inequality (see at the end of this section),by showing that u can be approximated by harmonic functions. Another way to prove it is by using interior derivative estimates for harmonic functions, which imply the equicontinuity on compact subdomains of *the second order derivatives* of any (uniformly) bounded collection of harmonic functions. Consequently by Arzela's theorem we have that any bounded sequence of harmonic functions forms a normal family. In order to see this, it is worth analysing the following example:

Example 10 ((([3],T2.10))**.** *From the mean value property and divergence theorems it follows*

$$Du(x) = \frac{1}{\omega_n R^n}\int_B Du\, dy = \frac{1}{\omega_n R^n}\int_{\partial B} u \cdot \nu\, ds. \tag{33}$$

One can show that if u is harmonic in a domain Ω and if Ω' is any compact subset of Ω and for any multi-index r with $d = \operatorname{dist}(\Omega', \partial\Omega)$ then following inequality holds

$$\sup_{\Omega'} |D^r u| \leq \left(\frac{n|r|}{d}\right)^{|r|} \sup_{\Omega} |u|. \tag{34}$$

Note that for any open convex subset of $\mathbb{R}^n$ we have the estimate

$$|u(x) - u(y)| \leq |Du(z)||x - y|. \tag{35}$$

Whenever the partial derivatives of u are bounded, u is Lipschitz continuous. More over, any bounded sequence of harmonic functions on a domain Ω contains a subsequence converging uniformly on compact subdomains of Ω to a continuous function. To show that this limit function is also a harmonic function, take any ball B contained in Ω, and for each element of the sequence, write the mean value property. Then the uniform convergence will imply that the mean value property is also satisfied by the limit function. Then conclude that ***any bounded sequence of harmonic*** *functions in a* *domain Ω contains a* ***subsequence converging uniformly on compact subdomains*** *of Ω to a* ***harmonic function****. (*[3]*, T2.11).*

Perron's method then ensures the existence of a solution to the equation $\Delta u = 0$ in a bounded domain Ω of $\mathbb{R}^n$, although it is sensible to expect that $u \leq \varphi$ on $\partial\Omega$. In the case that u is a solution of the *solvable Dirichlet problem* $\Delta u = 0$ in Ω, $u = \varphi$ on $\partial\Omega$, then this solution coincide with the solution given by Perron's method. However *the solvability of the Dirichlet problem depends on the geometric properties of the boundary* $\partial\Omega$. For each $z \in \partial\Omega$, we define the **barrier** at z relative to Ω, denoted by w_z, to be a $C^0(\bar{\Omega})$ function such that w_z is superharmonic in Ω, $w_z > 0 \in \bar{\Omega} \setminus \{z\}$ and $w_z(z) = 0$. A point $z \in \partial\Omega$ is called **regular** if there exists a barrier at z. It this at regular boundary points and for φ continuous at these, that the solution in Perron's method, u, matches with the boundary data of the Dirichlet problem, i.e., $\lim_{x\to z} u(x) = \varphi(z)$.

Theorem 11 (([3],T2.14))**.** *The classical Dirichlet problem with continuous boundary data is solvable if and only if the boundary points are all regular.*

The remaining problem is the characterisation of domains whose boundary points are all regular. In the case of $n = 2$, the Dirichlet problem is solvable if every component of the complement of the domain Ω has more than one element. Examples of this are domains bounded by a finite number of simple closed curves. In general dimension, when the boundary of Ω is C^2 then all boundary points are regular.

Now we turn to the Poisson's equation $\Delta u = f$ and state the existence result for the corresponding Dirichlet problem in a domain $\Omega \subset \mathbb{R}^n$. First we note that if f is continuous, then the **Newton potential** $\Gamma * f$ is not necessarily twice differentiable.

Example 12. *In $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1, x_2 > 0\}$, let $u(x_1, x_2) = x_1x_2 \log(x_1 + x_2)$ and $f(x_1, x_2) = 2\left(1 - \frac{x_1x_2}{(x_1+x_2)^2}\right)$. Then $\Delta u = f$, however $f \in L^\infty(\Omega)$ but $u \notin C^{1,1}(\Omega)$, since $u_{x_1x_2}$ is not bounded in Ω.*

Example 13. *For this it is necessary to assume f is Hölder continuous.*

Theorem 14 (([3], T4.3))**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and suppose that every boundary point is regular. Assume that f is bounded and locally Hölder continuous in Ω. Then for any continuous boundary values φ, the Dirichlet problem*

$$\left\{ \begin{array}{ll} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{array} \right. , \tag{36}$$

has a unique solution.

Proof. Note that if w is the Newtonian potential of f and if we consider $v = u - w$, the last problem is equivalent to Laplace's equation $\Delta v = 0$ in Ω and boundary values $\varphi - w$. □

The following is to address the question of the regularity of the solution after this limiting process. One can prove (see [3], L4.4) that if $B_1 \subset B_2$ are two concentric balls, $f \in C^\alpha(\bar{B}_2)$ where $0 \leq \alpha \leq 1$ then $w := \Gamma * f$ in B_2 is $C^{2,\alpha}$ in $\bar{B}_1$. One can improve this Hölder estimate to solutions of $\Delta u = f$, where u and f have compact support, so u belongs to $C^2_0(\mathbb{R}^n)$ and f in $C^\alpha_0(\mathbb{R}^n)$, implying that $u \in C^{2,\alpha}_0(\mathbb{R}^n)$ and more over:

Theorem 15 ([3], T4.6)**.** *Let $\Omega \subset \mathbb{R}^n$ a domain and suppose $u \in C^2(\Omega)$ and $f \in C^\alpha(\Omega)$ are such that $\Delta u = f$ in Ω. Then $u \in C^{2,\alpha}(\Omega)$. More over for any concentric balls $B_R \subset B_{2R} \subset\subset \Omega$ one can estimate*

$$|u|'_{C^{2,\alpha}(B_R)} \leq C(|u|_{C^0(B_{2R})} + R^2|f|'_{C^{0,\alpha}(B_{2R})}), \tag{37}$$

where $C = C(n, \alpha)$.

Last result gives the interior $C^{2,\alpha}$ regularity of solutions inside properly contained balls. See an application of the interior regularity for Linear operators below. As we have noted earlier, an immediate consequence of this interior estimate is the equicontinuity on compact subdomains of the *second derivatives* of any bounded family of solutions of the equation $\Delta u = f$. In the case of harmonic functions, interior estimates of derivatives and Arzela's Theorem is used to establish in [3],T2.11 that any bounded family of harmonic functions is a *normal family.* For the Poisson's equation then (37) and Arzela's theorem imply that any bounded family of solution to $\Delta u = f$ in Ω, with $f \in C^\alpha(\Omega)$, contains a subsequent converging uniformly on compact subdomain to a soluton. This compactness result gives the existence result for Poisson's equation:

Theorem 16 ([3]T4.9)**.** *Let B a ball in $\mathbb{R}^n$ and $f \in C^\alpha(B)$ such that*

$$\sup_{x\in B} d_x^{2-\beta}|f(x)| \leq N < \infty, \quad \textit{for some} \quad 0 < \beta < 1. \tag{38}$$

There there is a unique solution $u \in C^0(\bar{B}) \cap C^2(B)$, of the Dirichlet problem

$$\begin{cases} \Delta u = f & in \quad B \\ u = 0 & on \quad \partial B, \end{cases} \tag{39}$$

More over the following inequality holds

$$\sup_{x\in B_R} d_x^{-\beta}|u(x)| \leq CN, \ \textit{where the constant } C = C(\beta). \tag{40}$$

On the other hand, from Theorem 16 one can obtain a similar estimate replacing general domains Ω (not necessarily bounded, [3], T4.8). More over this estimate is also applicable to the intersection of the domain Ω and the upper half space $\mathbb{R}^n_+$ ([3], T4.11). The next result is about the regularity of the solutions of the Dirichlet problem in the ball, is states when a solution in $C^0(\bar{B}_R) \cap C^2(B_R)$ is also in $C^{2,\alpha}(\bar{B})$. The result follows from ([3], T4.11) applied to the Kelvin transformation.

Theorem 17 (Kellog's Theorem in the Ball ([3], T4.13, C4.14))**.** *Let $f \in C^\alpha(\bar{B})$ and $\varphi \in C^{2,\alpha}(\bar{B})$, and let $u \in C^0(\bar{B}) \cap C^2(B)$ a solution of the Dirichlet problem*

$$\begin{cases} \Delta u = f & in \quad B \\ u = \varphi & on \quad \partial B, \end{cases} \tag{41}$$

where $B \subset \mathbb{R}^n$ is a ball. Then $u \in C^{2,\alpha}(\bar{B})$, and the solution is unique.

Kellog's theorem is valid for more general domains $\Omega \subset \mathbb{R}^n$, and will be discussed in the following section.

Remark. Regarding the regularity of the solutions we will It is possible to prove using mollifiers that if u is a C^2 solution of the Laplace's equation, then u is smooth. Even more, if u is assumed to be continuous in the domain U and satisfies the mean value property for each ball inside the domain, then u is smooth, since its mollification is the same function. Recall that by a mollifier is a smooth function μ with the property that it can make sharped pieces of a given function to be *mollified* or smooth-out, after convolution. Some facts about mollifiers are: a) Re-scaling in ϵ - balls, the convolution of an ϵ-mollifier μ_ϵ with a locally integrable function is smooth. b) Almost everywhere, the point-wise convergence of $(\mu_\epsilon * f) \to f$ as $\epsilon \to 0$ holds. c) If f is continuous, the convergence is uniformly on compact subsets of the domain. d) if f is in L^p_{loc}, $1 \leq p < \infty$, the convergence is also in L^p_{loc} sense.

4.1. **Harnack's Inequality.** . (See [5]) If u is harmonic and non-negative in the Ball $B = B_R(0)$, then we have that for $x \in B$ and $y \in \partial B$ then

$$R - |x| = |\,|x| - |y|\,| \leq |x - y| \leq |x| + |y| = R + |x|,$$

this implies

$$\frac{R^2 - |x|^2}{nw_nR(R - |x|)^n} \geq \frac{R^2 - |x|^2}{nw_nR|y - x|^n} \geq \frac{R^2 - |x|^2}{nw_nR(R + |x|)^n},$$

or equivalently

$$\frac{R + |x|}{R - |x|}\frac{1}{nw_nR(R - |x|)^{n-2}} \geq \frac{R^2 - |x|^2}{nw_nR|y - x|^n} \geq \frac{R - |x|}{R + |x|}\frac{1}{nw_nR(R + |x|)^{n-2}},$$

after substituting in the Piosson's Integral Formula and a limit argument one can get the inequalities

$$\left(\frac{R}{R - |x|}\right)^{n-2}\frac{R + |x|}{R - |x|}u(0) \geq u(x) \geq \left(\frac{R}{R + |x|}\right)^{n-2}\frac{R - |x|}{R + |x|}u(0).$$

One consequence Harnack's inequality is that any harmonic function defined for all space and bounded from below is identically constant, this follows easily by adding a constant if necessary to have the function to be positive and by taking the limit as R tends to infinity.

Also it is possible to prove that if $\{u_n\}$ is a monotone increasing sequence of harmonic functions bounded from above at a point $p \in \Omega$, then in each compact subset of Ω, $\{u_n\}$ converges uniformly to a harmonic function.

A generalisation of this convergence result for uniform limit of solutions to a linear second order PDE will be a consequence of the Schauder's interior estimates, to be discuss in the following section.

5. Linear Second Order Partial Differential Equations: Schauder Theory.

There exists a complete theory for linear equations relying in a priori estimates. These are inequalities on the values of solutions and its derivatives, before actually guarantee its existence.

By solving the Poisson's equation $\Delta u = f$ in a domain $\Omega \subset \mathbb{R}^n$ it is possible to show that for any subset $\Omega' \subset\subset \Omega$, if $u \in C^2(\Omega)$ is a solution of the equation, then

$$||u||_{C^{2,\alpha}(\bar{\Omega}')} \leq C(||u||_{C^0(\Omega)} + ||f||_{C^\alpha(\bar{\Omega})}), \tag{42}$$

with $C = C(\alpha, n, d(\Omega', \partial\Omega))$, where $\alpha \in (0,1)$ and $d(\Omega', \partial\Omega)$ is the distance between the sets. This estimate can be turned into a global estimate for solutions with sufficiently smooth boundary values, and with boundary $\partial\Omega$ sufficiently smooth.

Schauder theory consists in obtaining the same inequality for any $C^{2,\alpha}$ solution of $Lu = f$, where now the constant C depends additionally on bounds of the coefficients in Hölder space sense and also on the minimum and maximum eigenvalues of of the coefficient matrix (a^{ij}) in Ω.

This estimates are used to apply the Continuity method and obtain the existence of a solution for the Dirichlet from perturbations of the Poisson's equation.

For solutions of elliptic linear equations in bounded domains, the maximum principle gives the following point wise estimate

Theorem 18 ([3],T3.6)**.** *Let $Lu = f$ in a bounded domain $\Omega \subset \mathbb{R}^n$ where L is elliptic, $c \leq 0$ and $u \in C^0(\Omega) \cap C^2(\Omega)$ is a solution. Then*

$$|u|_{C^0(\Omega)} \leq \sup_{\partial\Omega} |u| + C \sup_{\Omega} \frac{|f|}{\lambda}, \tag{43}$$

where $C = C(\operatorname{diam}\Omega, \beta = \sup|b|/\lambda)$.

Theorem 19 (Schauder's estimate,([3],T6.6))**.** *Let $\Omega \subset \mathbb{R}^n$ be a $C^{2,\alpha}$ domain and assume $u \in C^{2,\alpha}(\bar{\Omega})$ is a solution of $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$, where $\varphi \in C^{2,\alpha}(\bar{\Omega})$, $f \in C^\alpha(\bar{\Omega})$, L is uniformly elliptic. Also assume that there are positive constants λ, Λ such that for $a^{ij}, b^i, c \in C^\alpha(\Omega)$ we have*

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad x \in \Omega, \xi \in \mathbb{R}^n, \tag{44}$$

and

$$|a^{ij}|_{C^\alpha(\Omega)}, |b^i|_{C^\alpha(\Omega)}, |c|_{C^\alpha(\Omega)} \leq \Lambda. \tag{45}$$

Then there is a constant $C = C(n, \alpha, \lambda, \Lambda, \Omega)$ such that the following inequality holds

$$|u|_{C^{2,\alpha}(\Omega)} \leq C(|u|_{C^0(\Omega)} + |\varphi|_{C^{2,\alpha}(\Omega)} + |f|_{C^\alpha(\Omega)}). \tag{46}$$

From this a priori estimate then we can start with a solution for the Poisson's equation $\Delta u = f$ and then trace a solution for $Lu = f$ through solutions of a family of equations joining $\Delta u = f$ with $Lu = f$. This idea is stated in the following theorem.

Theorem 20 (Continuity Method ([3],T5.2))**.** *Let B be a Banach space, V a normed linear space, and $L_0, L_1 : B \to V$ two bounded linear operators. Define for $0 \leq t \leq 1$ the family*

$$L_t = (1-t)L_0 + tL_1, \tag{47}$$

and assume there exists a constant C such that

$$|x|_B \leq C|L_t x|_V, \quad 0 \leq t \leq 1. \tag{48}$$

Then L_1 is surjective if and only if L_0 is surjective.

First we are going to stablish the existence for the Dirichlet problem in th case where Ω is the ball.

Proposition 21 (([3], C6.9))**.** *Let B a ball in $\mathbb{R}^n$ and L strictly elliptic in B, with coefficients in $C^\alpha(\bar{B})$ and $c \le 0$. Then if $f \in C^\alpha(\bar{B})$ and $\varphi \in C^{2,\alpha}(\bar{B})$, there is a unique solution $u \in C^{2,\alpha}(\bar{B})$ of the boundary value problem $Lu = f$ in B, $u = \varphi$ on ∂B.*

Proof. The detailed proof follows the one given below for more general domains Ω and using Kellogg's theorem on the ball (Theorem 17). □

Now we are going to stablish the Perron's method for linear equations. We introduce the concept of **subsolution** of $Lu = f$, namely a funciton $u \in C^0(\Omega)$ such that for every ball B properly contained in Ω and every solution v of $Lv = f$ in B, if $u \le v$ on ∂B then $u \le v$ in B. By changing $\le$ by $\ge$, we say that $u \in C^0(\Omega)$ is a **supersolution** of $Lu = f$. When $f \in C^\alpha(\Omega)$ and also the coefficients of L are Hölder continuous, $c \le 0$ and such that the maximum principle holds, then we have the following properties:

- The function $u \in C^2(\Omega)$ is a subsolution if and only if $Lu \ge f$.
- Let Ω be bounded, u and v a subsolution and a supersolution in Ω respectively. If $v \ge u$ on ∂B then either $v \ge u$ in Ω or they are equal.
- Let B a ball such that $\bar{B} \subset \Omega$ and u a subsolution in Ω. Let $\bar{u}$ the solution to the Dirichlet problem in the ball $L\bar{u} = f$ in B such that $\bar{u} = u$ on ∂B. Then we can obtain a subsolution U in Ω given by
$$U(x) = \begin{cases} \bar{u}(x) & x \in B \\ u(x) & x \in \Omega \setminus B \end{cases}.$$
- Let $u_1, \ldots, u_k$ be subsolutions in Ω. Then the function $u(x) = \max_i\{u_i(x)\}$ is a subsolution in Ω.

Let Ω be bounded and φ a bounded function on $\partial\Omega$. We say that $u \in C^0(\bar{\Omega})$ is a **subfunciton** relative to φ if u is a subsolution in Ω and $u \le \varphi$ on $\partial\Omega$. We say that u is a **supersolution** relative to φ if it is a superfunction relative to φ in Ω and $u \ge \varphi$ on $\partial\Omega$. Define S_φ to be the set of all subfunctions in Ω relative to φ. In the case L is strictly elliptic in Ω and f and the coefficients of L are bounded, then S_φ is non-empty and bounded from above.

Theorem 22 (Perron's process, ([3], T6.11))**.** *Let Ω a bounded domain, L a strictly elliptic operator with $c \le 0$ and coefficients in $C^\alpha(\Omega)$, the funciton $f \in C^\alpha(\Omega)$, and φ a bounded function on $\partial\Omega$. Then the function $u(x) = \sup_{v \in S_\varphi} v(x)$ belongs to $C^{2,\alpha}(\Omega)$ and is such that $Lu = f$ in Ω whenever u is bounded.*

Next we analyse the conditions that will make this soulution to assume given boundary values. Let φ bounded on $\partial\Omega$ and continuous at $x_0 \in \partial\Omega$. The sequence of functions $\{w_i^+\}$ and $\{w_i^-\}$ are called respectively **upper barrier** and **lower barrier** in Ω relative to L, f and φ at the point $x_0 \in \Omega$ if each w_i^+ is a superfunction relative to φ in Ω, w_i^- is a subfunction relative to φ in Ω, and $\lim_{i\to\infty} w_i^\pm(x_0) = \varphi(x_0)$. If at a point there exist both upper and lower barrier, then we say simply that there is a barrier at that point.

Theorem 23 (([3], L6.12))**.** *For a bounded function φ on $\partial\Omega$ continuous at $x_0 \in \partial\Omega$, the solution of the Dirichlet problem $Lu = f$ in Ω given by the Perron's process, satisfy the boundary condition $\lim_{x\to x_0} u(x) = \varphi(x_0)$ if there exist a barrier at x_0.*

Again the question of what domains admit a barrier is of interest. In particular, any C^2 domain, and domains satisfying an exterior sphere condition at every point on the boundary has a barrier ([3],T6.13).

Now, with Perron's process we can now prove a more general version of Kellogg's Theorem.

Theorem 24 (Kellogg's Theorem, ([3], T6.14))**.** *Let L be strictly elliptic in a bounded domain Ω with $c \le 0$ and let f and the coefficients of L belong to $C^\alpha(\bar{\Omega})$. Suppose that Ω is a $C^{2,\alpha}$ domain and that $\varphi \in C^{2,\alpha}(\bar{\Omega})$. If $u \in C^0(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$ is a solution of the Dirichlet problem $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$. Then $u \in C^{2,\alpha}(\bar{\Omega})$.*

Proof. Since Ω is in particular a C^2 domain we apply Theorem 23. There exist then a solution $u \in C^0(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$. Then we have to check only that at any point $x_0 \in \partial\Omega$ there is a neighbourhood V of x_0 such that $u \in C^{2,\alpha}(V \cap \bar{\Omega})$. □

Now we get the general existence theorem for strictly linear equations in $C^{2,\alpha}$ domains.

Theorem 25 (([3], T6.8)). *Let $\Omega \subset \mathbb{R}^n$ be a $C^{2,\alpha}$ domain and L a strictly elliptic linear operator in Ω with coefficients in $C^\alpha(\bar{\Omega})$, with $c \leq 0$. Assume that the Dirichlet problem for Poisson's equation $\Delta u = f$ in Ω, $u = \varphi$ on $\partial\Omega$ has a $C^{2,\alpha}(\bar{\Omega})$ solution for all $f \in C^\alpha(\bar{\Omega})$ and all $\varphi \in C^{2,\alpha}(\bar{\Omega})$, then the Dirichlet problem*

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned} \tag{49}$$

also has a unique solution in $C^{2,\alpha}(\bar{\Omega})$ for all such f and φ.

Proof. Note that (by linearity) we can restrict to the case of zero boundary values: $Lv = g$ in Ω and $v = 0$ on $\partial\Omega$ (take $v = u - \varphi$, and $g = f - L\varphi$).

Now the following family of equations is considered

$$L_t = (1-t)\Delta + tL, \quad 0 \leq t \leq 1, \tag{50}$$

where from the hypothesis on the coefficients of L we can see that L_t satisfies

$$\begin{aligned} a_t^{ij}(x)\xi_i\xi_j &\geq \lambda_t|\xi|^2, \quad \forall\, x \in \Omega,\ \xi \in \mathbb{R}^n, \\ |a_t^{ij}|_{C^\alpha(\Omega)}, |b_t^i|_{C^\alpha(\Omega)}, |c_t|_{C^\alpha(\Omega)} &\leq \Lambda_t. \end{aligned} \tag{51}$$

These bounds are independent of t when we take

$$\lambda_t = \min(1,\lambda), \quad \Lambda_t = \max(1,\Lambda). \tag{52}$$

Let $B_1 = \{u \in C^{2,\alpha}(\Omega) : u = 0, \text{on}\,\partial\Omega\}$ and $B_2 = C^\alpha(\bar{\Omega})$. Note then that $L_t : B_1 \to B_2$ is a bounded linear operator between Banach spaces. Let u_t a solution of $L_t u = f$ for arbitrary $f \in C^\alpha(\Omega)$. From maximum principle, we obtained Theorem 18, and in this case

$$|u_t|_{C^0(\Omega)} \leq C|f|_{C^\alpha(\Omega)}. \tag{53}$$

Additionally from equation (46) in Theorem 19 we have

$$|u_t|_{C^{2,\alpha}(\Omega)} \leq C(|u|_{C^0(\Omega)} + |f|_{C^\alpha(\Omega)}) = C|f|_{C^\alpha(\Omega)}, \tag{54}$$

equivallently we have

$$|u|_{B_1} \leq C|L_t u|_{B_2}, \tag{55}$$

with C is independent of t. Kellog's theorem states that the solution u of the Dirichlet problem for the Poisson's equation belongs to $C^{2,\alpha}(\bar{\Omega})$ for $f \in C^\alpha(\Omega)$ and when Ω and the boundary values are also $C^{2,\alpha}$. Then we have $L_0 = \Delta : B_1 \to B_2$ is onto and by continuity method L_1 is invertible. □

To investigate higher regularity of solutions, the following lemma states that if u is a $C^2(\Omega)$ solution of $Lu = f$, and if f and the coefficients of L are in C^α, then u should also be in $C^{2,\alpha}$.

Lemma 26 (([3], L6.16)). *Let $u \in C^2(\Omega)$ be a solution of the equation $Lu = f$, in an open set Ω, where f and the coeffients of the elliptic operator L are in $C^\alpha(\Omega)$. Then $u \in C^{2,\alpha}(\Omega)$.*

The previous lemma and Schauder estimates gives the following

Theorem 27 (Interior Regularity Theorem ([3], T6.17)). *Let $u \in C^2(\Omega)$ be a solution of the equation $Lu = f$ in the open set Ω, where f and the coefficients of the elliptic operator L are in $C^{j,\alpha}(\Omega)$. Then $u \in C^{2+j,\alpha}(\Omega)$.*

Proof. For $j = 0$, we have the previous lemma. Let $j = 1$, so f and the coefficients of L are in $C^{1,\alpha}(\Omega)$, and assume $u \in C^2(\Omega)$. Hence $u \in C^{2,\alpha}(\Omega)$. Define for $h \in \mathbb{R}$ small, and $e_1 = (1,0,\ldots,0) \in \mathbb{R}^n$ the following quotient

$$u^h := h^{-1}\left(u(x+he_1) - u(x)\right).$$

Note that

$$Lu^h = a_{ij}u^h_{ij} + b_iu^h_i + cu^h = f^h - a^h_{ij}u_{ij}(x+he_1) - b^h_iu_i(x+he_1) - c^hu(x+he_1).$$

The fact that $f \in C^{1,\alpha}(\Omega)$ and the following integral identity

$$f^h(x) = \frac{1}{h}\int_0^1 \frac{d}{dt}f(x+the_1)dt = \int_0^1 \frac{\partial}{\partial x_1}f(x+the_1)dt,$$

imply that $f^h \in C^\alpha(\Omega)$. Same conclusion applies for the coefficients a^h_{ij}, b^h_i, c^h. Using convergence arguments and the fact that $u \in C^{2,\alpha}(\Omega)$ implies that $u^h \in C^{2,\alpha}$ and that $\lim_{h\to 0} u(x+he_1) = u(x)$ is actually $C^{3,\alpha}$. The rest of the proof follows by induction. □

More over, it is possible to obtain also the global regularity theorem using a stronger version of the lemma above and have a priori estimates up to the boundary

Theorem 28 (Global Regularity Theorem ([3], T6.19)). *Let Ω be a $C^{j,\alpha}$ domain, $j \geq 0$, and $\varphi \in C^{k+2,\alpha}(\bar{\Omega})$. Let $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ be a solution of the Dirichlet problem $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$, where f and the coefficients of the strictly (uniformly) elliptic operator L are in $C^{j,\alpha}(\bar{\Omega})$. Then $u \in C^{2+j,\alpha}(\bar{\Omega})$.*

6. Quasi-linear Second Order Partial Differential Equations: The Leray-Schauder Fixed Point Theorem.

Following the same spirit of the last section, here we consider the Dirichlet problem $Qu = 0$ in Ω, $u = \varphi$ on $\partial\Omega$ such that Ω is a bounded domain, the coefficients of Q are C^α in their domain, the values on the boundary $\varphi \in C^{2,\alpha}(\bar{\Omega})$. Then the existence of a solution $u \in C^{2,\alpha}(\bar{\Omega})$ if we can stablish the a priori estimate

$$|u|_{C^{1,\beta}(\bar{\Omega})} \leq M. \tag{56}$$

A four-step process is stated in [3] to get this estimate is the successive estimation of $\sup_\Omega |u|$ (using maximum principle for quasilinear equations), $\sup_{\partial\Omega} |Du|$, $\sup_\Omega |Du|$, each using the preceding ones, and finally obtain an estimate for $|u|_{C^{1,\beta}(\bar{\Omega})}$ for some $\beta > 0$.

There are lots of results regarding the gradient estimates in the interior and in the boundary. It is possible to obtain interior estimates in terms of the gradient estimate on the boundary by a refinement of the so called Bernstein technique. This is precisely

The existence result is obtained by an extension of the Brouwer fixed point theorem:

Theorem 29. *Let $B \subset \mathbb{R}^n$ be an open ball and $T : \bar{B} \to \bar{B}$ a continuous map. Then there is some $x \in \bar{B}$ such that $T(x) = x$, i.e, T has at least one fixed point.*

Theorem 30 ([3], T11.3, Leray-Schauder). *Let B a Banach space and $T : B \to B$ a compact mapping, that is, the image under T of any bounded set have compact closure. Suppose that there exists $M > 0$ such that*

$$|x|_B < M \tag{57}$$

for all $x \in B$. Then for all $\sigma \in [0,1]$, σT has a fixed point.

Now we apply this result to establish the existence of a solution for the quasi-linear case.

Theorem 31 ([3], T11.4). *Let $\Omega \in \mathbb{R}^n$ be a bounded domain and suppose that the quasilinear equation Q is elliptic in $\bar{\Omega}$ with Hölder continuous coefficients $a^{ij}, b \in C^\alpha(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$, $0 \leq \alpha \leq 1$, the boundary $\partial\Omega \in C^{2,\alpha}$ and assume that the boundary values $\varphi \in C^{2,\alpha}(\bar{\Omega})$. Consider the following one parameter family of boundary value problems*

$$\begin{cases} Q_\sigma u = 0 & in \quad \Omega \\ u = \sigma\varphi & on \quad \partial\Omega, \end{cases} \tag{58}$$

where Q_σ is defined for $0 \leq \sigma \leq 1$ by

$$Q_\sigma u = a^{ij}(x, u, Du)D_{ij}u + \sigma b(x, u, Du). \tag{59}$$

If for some $\beta > 0$ there is a constant M independent of u and σ, such that for every solution $u \in C^{2,\alpha}(\bar{\Omega})$ of the Dirichlet problem $Q_\sigma u = 0$ in Ω, $u = \sigma\varphi$ on $\partial\Omega$, and this constant satisfies

$$|u|_{C^{1,\beta}(\bar{\Omega})} < M, \tag{60}$$

then the Dirichlet problem $Qu = 0$ in Ω, $u = \varphi$ on $\partial\Omega$, is solvable in $C^{2,\alpha}(\bar{\Omega})$.

Proof. Recall that if $0 < \alpha \leq \beta < 1$, and Ω is a bounded set, then for every $f \in C^\beta(\Omega)$ we have

$$|f|_{C^\alpha(\Omega)} \leq \operatorname{diam}\Omega^{\beta-\alpha}|f|_{C^\beta(\Omega)}, \tag{61}$$

from where the inclusion $C^\beta(\Omega) \to C^\alpha(\Omega)$ follows. This inclusion is also compact as consequence of Arzela-Ascoli theorem. Then there is a compact inclusion of $C^{2,\alpha\beta}(\bar{\Omega})$ in $C^{1,\beta}(\bar{\Omega})$ and another in $C^{2,\alpha}(\bar{\Omega})$.

Define the operator $T : C^{1,\alpha}(\bar{\Omega}) \to C^{2,\alpha\beta}(\bar{\Omega})$ by letting $Tv = u$ be the unique solution in $C^{2,\alpha\beta}(\bar{\Omega})$ of the linear Dirichlet problem,

$$\begin{aligned} Qu = a^{ij}(x, v, Dv)D_{ij}u + b(x, v, Dv) = 0, \text{ in } \Omega \\ u = \varphi, \text{ on } \partial\Omega, \end{aligned} \tag{62}$$

where we are applying the existence and uniqueness result from the theory of elliptic linear equations. Then a soulution in $u \in C^{2,\alpha}(\bar{\Omega})$, is a fixed point of T, for a solution of the Dirichlet problem above is actually a solution of the equation $Tu = u$ in $C^{1,\beta}(\bar{\Omega})$.

Note on the other hand that the equation $\sigma Tu = u$ in $C^{1,\beta}(\bar{\Omega})$, is equivalent to the Dirichlet problem

$$\begin{aligned} Q_\sigma u = a^{ij}(x, u, Du)D_{ij}u + \sigma b(x, u, Du) = 0, \text{ in } \Omega \\ u = \sigma\varphi, \text{ on } \partial\Omega, \end{aligned} \tag{63}$$

We have to show the compactness and continuity of the operator T in order to apply the Leray-Schauder theorem. That T is a compact mapping follows from the fact that any bounded set in $A \subset C^{1,\beta}(\bar{\Omega})$, then by Schauder's estimate, T maps A into a bounded set in $C^{2,\alpha\beta}(\bar{\Omega})$. By Arzela's Theorem, this is precompact in $C^2(\bar{\Omega})$ and $C^{1,\beta}(\bar{\Omega})$.

Continuity of T we take a convergent sequence $\lim v_m = v$ in $C^{1,\beta}(\bar{\Omega})$. Note that Tv_m is precompact in $C^2(\bar{\Omega})$. This means that for every subsequence of $\{v_m\}$, the corresponding $\{Tv_m\}$ has convergent subsequence. Call $\{T\bar{v}_m\}$ such a convergent subsequence and let $u \in C^2(\bar{\Omega})$ be its limit. Since

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\{a^{ij}(x, \bar{v}_m, D\bar{v}_m)D_{ij}T\bar{v}_m + b(x, \bar{v}_m, D\bar{v}_m)\right\} \\ &= a^{ij}(x, v, Dv)D_{ij}u + b(x, v, Dv), \end{aligned} \tag{64}$$

then we conclude that $Tv = u$ and then the sequence Tv_m converges to u. □

7. Fully Nonlinear Partial Differential Equation

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We are concern in operators of the type

$$F[u] = F(x, u, Du, D^2u) \tag{65}$$

where $\Omega \subset \mathbb{R}^n$ is an open domain, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, Du denote the gradient of u and D^2u the hessian matrix. F is then a function on $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$, where $\mathbb{R}^{n\times n}$ denotes the space of real symmetric $n \times n$ matrices and write $\gamma = (x, z, p, r) \in \Gamma$. If the function F is an affine function with respect to the r variables then we say that F is *quasilinear*, and in any other case we say that F is *fully nonlinear*.

Definition 32. *The operator F is called* ***elliptic*** *in $\mathcal{U} \subset \Gamma$ if the matrix*

$$F_{ij}(\gamma) = \frac{\partial F}{\partial r_{ij}}(\gamma) \tag{66}$$

is positive. If $\lambda(x)$ and $\Lambda(x)$ denote respectively the minimum and maximum eigenvalue of $[F_{ij}(x)]$, then F is called ***uniformly elliptic*** *if Λ/λ is bounded and* ***strictly elliptic*** *if $1/\lambda$ is bounded.*

Example 33. *One can see that in dimension $n = 2$, the ellipticity of $F = F(x, u, Du, D^2u)$ is equivalent to have*

$$4F_{11}F_{22} - F_{12} > 0. \tag{67}$$

One example to have in mind is the Monge-Ampére equation

$$\det(D^2u) = f \quad \text{in } \Omega. \tag{68}$$

It turns out to be elliptic in the class of convex functions u, and then necessarily $f > 0$.

We will see in this section that by the continuity method, the existence of a solution for the Dirichlet problem is reduced to obtain the a priori estimate

$$|u|_{C^{2,\alpha}(\bar{\Omega})} \leq C, \tag{69}$$

for some $0 < \alpha < 1$. Then, like in the quasilinear case we have to establish estimates for $\sup_\Omega |u|$, $\sup_{\partial\Omega} |Du|$, $\sup_\Omega |Du|$, and additionally $\sup_{\partial\Omega} |D^2u|$, $\sup_\Omega |D^2u|$.

Int he case F is an uniformly elliptic, fully nonlinear concave equation, the Evans-Krylov theorem [1][4], gives the following a priori estimate

$$|u|_{C^{2,\alpha}(\bar{\Omega})} \leq C|u|_{C^{1,1}(\bar{\Omega})}, \tag{70}$$

when for the case $\Omega = B_1$ is the unit ball and C depends only on the concavity of F.

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